

Relativistically induced divergence of electric field

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It is demonstrated that, divergence of electric field can vary due to relativistic transformation, i.e. divergence of electric field without net charge may be observed around a straight wire in certain moving frame. On the other hand, the divergence of magnetic field remains invariantly zero, though within specific region, magnetic field can appear as if it was around magnetic charge when field of a solenoid is viewed from a moving frame. It indicates that divergence of electric field may not be exclusively associated with electric charges, thus violates the Gauss law of Maxwell equations.

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In general, vector fields have divergence and curl. In electrodynamics, the divergence of field has been known to be associated only with electric charges. An interesting attempt to symmetrize electricity and magnetism were the introduction of magnetic charges[1, 2], which have been extensively studied in both theories[3–6] and experiments[7–12] because it explains the quantization of electric charges.

Although it has been observed that electric charge is invariant[13], we reveal a model in which divergence of electric field may be observed due to relativistic effect. Our scheme is based on the observation that Lorentz transformation may cause electric and magnetic field interchange, at the same time, field components with looped field lines may induce radial field components when viewed from certain moving frame. Analysis of these observed radial fields shows that, 1) divergence of electric field can vary in different observing frame, 2) on the other hand, divergence of magnetic field remains invariantly zero, though in specific region, magnetic field can appear as if it was around magnetic charge. Comparison of 2 setups reveals why divergence of electric field can be varying in different observing frame while no divergence of magnetic field can result from the same mechanism.

In Gaussian unit, the transformation of electromagnetic fields from reference frame F to frame F' moving with velocity $\mathbf{v}$ relative to F are[14]

$$\begin{aligned} [\mathbf{E}']_{\mathbf{x}',t'} &= [\gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E})]_{\mathbf{x},t} \\ [\mathbf{B}']_{\mathbf{x}',t'} &= [\gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B})]_{\mathbf{x},t} \\ \boldsymbol{\beta} &= \frac{\mathbf{v}}{c}, \beta = |\boldsymbol{\beta}|, \gamma = \frac{1}{\sqrt{1-\beta^2}} \end{aligned} \quad (1)$$

which, in SI unit, becomes

$$\begin{aligned} [\mathbf{E}']_{\mathbf{x}',t'} &= [\gamma(\mathbf{E} + c\boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E})]_{\mathbf{x},t} \\ [\mathbf{B}']_{\mathbf{x}',t'} &= [\gamma(\mathbf{B} - \frac{\boldsymbol{\beta}}{c} \times \mathbf{E}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B})]_{\mathbf{x},t} \end{aligned} \quad (2)$$

The subscripts indicate that the left hand side is field at $(\mathbf{x}', t')$ in frame F' , while the right hand side contains

fields at $(\mathbf{x}, t)$ in frame F . $(\mathbf{x}', t')$ and $(\mathbf{x}, t)$ are related by Lorentz transformation

$$\begin{aligned} ct' &= \gamma(ct - \boldsymbol{\beta} \cdot \mathbf{x}) \\ \mathbf{x}' &= \mathbf{x} + \frac{\gamma-1}{\beta^2}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{x}) - ct\gamma\boldsymbol{\beta} \end{aligned} \quad (3)$$

Geometrically, the force lines of field with nonvanishing curl are closed loops, while fields with nonvanishing divergence has radial force lines. In eqn. (1), the terms $\gamma\boldsymbol{\beta} \times \mathbf{B}$ and $-\gamma\boldsymbol{\beta} \times \mathbf{E}$ show interchanges between electric field and magnetic field. They also represent transformations between azimuthal component and radial component of the fields, depicted in fig. 1. As to be shown, the radial electric field $\mathbf{E}'$ in fig. 1b has varying divergence in different observing frame, while in fig. 1c, the radial magnetic induction $\mathbf{B}'$ outside specific region is identical to field that could be around magnetic charge, though the total divergence remains invariantly zero.

Following are studies of the relativistically induced radial field components in 2 specific setups corresponding to fig. 1b and 1c respectively. One deals with an infinite long ideal solenoid, the other deals with the fields around an infinite long current carrying straight wire.

An infinite long ideal solenoid can form a realization of the field transformation shown in fig. 1c. To proceed, we first show how a quasi static rotational electric field can emerge from Maxwell's equation, the wave equation are

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} &= \frac{1}{\epsilon_0 c^2} \frac{\partial \mathbf{J}}{\partial t} \\ \nabla \times \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -\mu_0 \nabla \times \mathbf{J} \end{aligned} \quad (4)$$

Different configuration of $\mathbf{J}$ can result in a variety of field structures $(\mathbf{E}, \mathbf{B})$. For our purpose, we take

$$\mathbf{J}(\mathbf{r}, t) = t\mathbf{j}(\mathbf{r}), \nabla \cdot \mathbf{j} = 0 \quad (5)$$

then the wave equation becomes

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} &= \frac{1}{\epsilon_0 c^2} \mathbf{j} \\ \nabla \times \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -\mu_0 t \nabla \times \mathbf{j} \end{aligned} \quad (6)$$

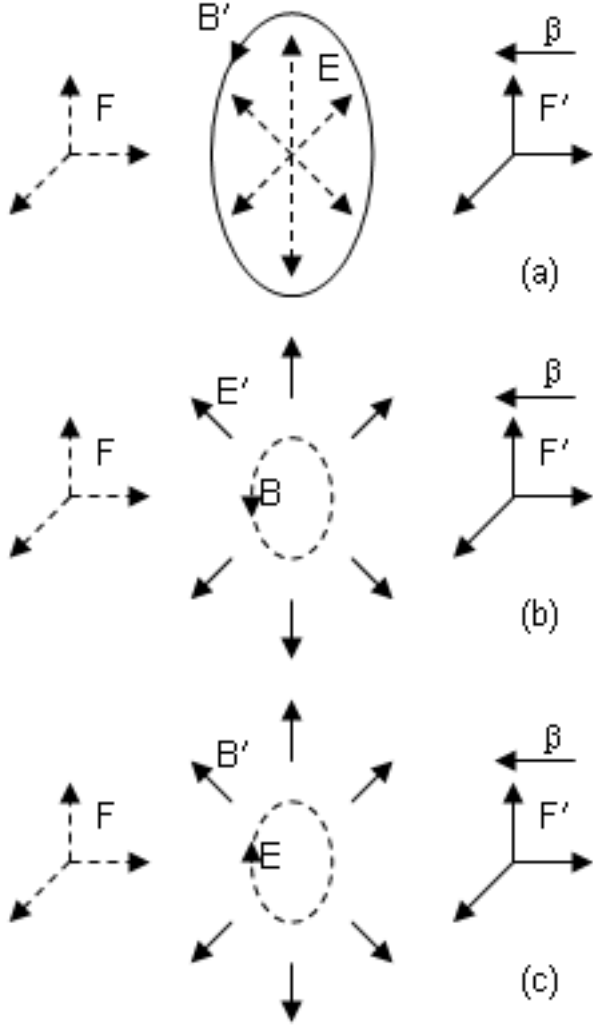


FIG. 1: Dashed lines are fields observed in frame F , solid lines are fields observed in frame F' moving with velocity β relative to frame F . (a) Radial component of field $\mathbf{E}$ induces circular shaped $\mathbf{B}'$. (b) β and $\mathbf{B}$ in frame F form a left hand screw, circular shaped $\mathbf{B}$ induces outward pointing radial $\mathbf{E}' = \gamma\beta \times \mathbf{B}$ in frame F' . (c) β and $\mathbf{E}$ in frame F form a right hand screw, circular shaped $\mathbf{E}$ induces outward pointing radial $\mathbf{B}' = -\gamma\beta \times \mathbf{E}$ in frame F' .

In the first equation, the right hand side is time independent, so a static electric field $\mathbf{E}(\mathbf{r}, t) = \mathbf{e}(\mathbf{r})$ may be possible. In the second equation, the right hand side is depending on time linearly and the left hand side contains a second order derivative with respect to t , so magnetic induction in the form $\mathbf{B}(\mathbf{r}, t) = t\mathbf{b}(\mathbf{r})$ may well be the solution. Then with driving source $\mathbf{J}$ of eqn. (5), we may write the solution of wave equation (4) as

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= \mathbf{e}(\mathbf{r}) \\ \mathbf{B}(\mathbf{r}, t) &= t\mathbf{b}(\mathbf{r})\end{aligned}\quad (7)$$

thus fields $\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)$ are represented as product of time factors and spatial factors. Put eqn. (7) into eqn.

(6), $\mathbf{e}(\mathbf{r}), \mathbf{b}(\mathbf{r})$ should satisfy

$$\begin{aligned}\nabla \times \nabla \times \mathbf{e} &= \frac{1}{\epsilon_0 c^2} \mathbf{j} \\ \nabla \times \nabla \times \mathbf{b} &= -\mu_0 \nabla \times \mathbf{j}\end{aligned}\quad (8)$$

the time factors disappear. It can further be written

$$\begin{aligned}\nabla \times \mathbf{e} &= -\mathbf{b} \\ \nabla \times \mathbf{b} &= -\mu_0 \mathbf{j}\end{aligned}\quad (9)$$

Quantities involved here $(\mathbf{b}, \mathbf{e}, \mathbf{j})$ are only the spatial factors of the true fields $(\mathbf{B}, \mathbf{E}, \mathbf{J})$ and are static. Combining with eqn. (7), eqn. (9) means, a driving source of linearly varying current generates a quasi static field, which contains a static electric field and a linearly varying magnetic induction. The spatial distribution of $\mathbf{B}$ is the same as the one with constant current source.

We notice, the linearly varying magnetic field $\mathbf{B}(\mathbf{r}, t) = t\mathbf{b}(\mathbf{r})$ creates magnetic displacement current J_m , just like the electric displacement current. And the static electric field around this magnetic displacement current has non-vanishing curl.

For an infinite long solenoid with circular cross section of radius R_0 and N coils per unit length, and carrying a static current $I(t) = I_0$, the electric field is 0 everywhere, the magnetic field inside the solenoid is axial and the amplitude is

$$|B| = \begin{cases} 0, & R > R_0 \\ \mu_0 N I_0, & R < R_0 \end{cases}\quad (10)$$

where R is the perpendicular distance from the observation point to the axis of the solenoid.

For an infinite long solenoid with circular cross section of radius R_0 and N coils per unit length, carrying linearly varying current $I(t) = tI_0$, the field can be computed from eqn. (7)-(10), there is a linearly varying axial magnetic induction $\mathbf{B}$ inside the solenoid,

$$|B| = \begin{cases} 0, & R > R_0 \\ t\mu_0 N I_0, & R < R_0 \end{cases}\quad (11)$$

and there is an azimuthal electric field shown in fig. 2 as dashed line, the magnitude is

$$|E| = \begin{cases} \frac{\mu_0}{2R} R_0^2 N I_0, & R > R_0 \\ \frac{\mu_0 R}{2} N I_0, & R < R_0 \end{cases}\quad (12)$$

When one takes measurement from frame F' moving along z axis at velocity of β as shown in fig. 3, radial component of the magnetic induction $\mathbf{B}'$ can be observed. By eqn. (2), the observed $\mathbf{B}'$ in frame F' contains an $\mathbf{E}$ related component $\mathbf{B}'_r$ and a $\mathbf{B}$ related component $\mathbf{B}'_z$

$$\begin{aligned}[\mathbf{B}'_r]_{\mathbf{x}', t'} &= [-\frac{\gamma\beta}{c} \times \mathbf{E}]_{\mathbf{x}, t} \\ [\mathbf{B}'_z]_{\mathbf{x}', t'} &= [\gamma\mathbf{B} - \frac{\gamma^2}{\gamma+1}\beta(\beta \cdot \mathbf{B})]_{\mathbf{x}, t}\end{aligned}\quad (13)$$

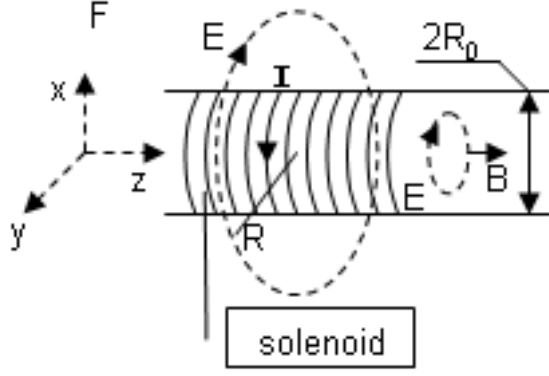


FIG. 2: Around a solenoid carrying a linearly growing current $I = tI_0$, axial magnetic induction $\mathbf{B} = t\mathbf{b}$ exists only inside the solenoid and grows linearly, and there is quasi static electric field whose line is concentric circular.

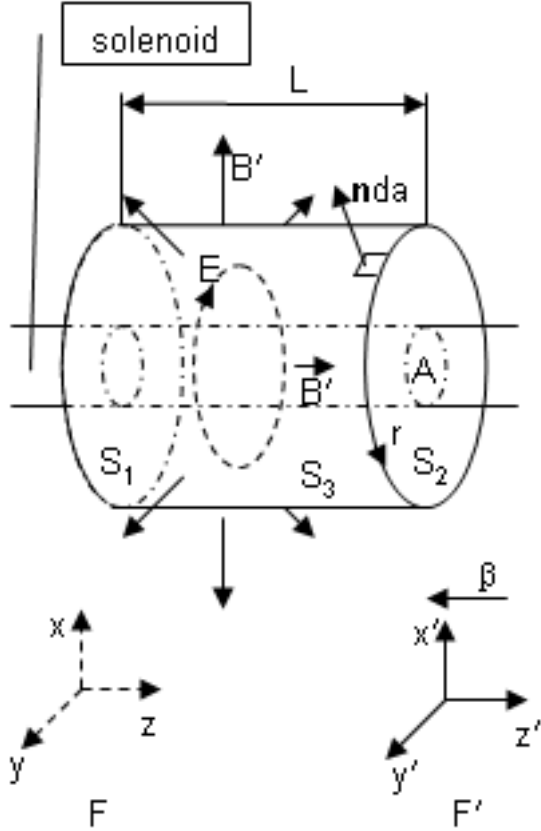


FIG. 3: For the solenoid in fig. 2, magnetic induction $\mathbf{B}'$ observed in axially moving frame F' and a closed cylindrical surface around the solenoid

with the setup of solenoid shown in fig. 3, $\mathbf{B}$ has only z component in frame F , so $\mathbf{B} = B_z \hat{z}$, $[\mathbf{B}'_r]_{x',t'}$ is radial, while $[\mathbf{B}'_z]_{x',t'}$ is axial and exists only within the solenoid as $\mathbf{B} = B_z \hat{z}$ vanishes outside the solenoid. The magnitude

of the radial field is

$$|B'_r| = -\frac{\gamma\beta}{c}|E| = \begin{cases} -\frac{\gamma\beta}{c}\frac{\mu_0}{2}R_0^2NI_0, & R > R_0 \\ -\frac{\gamma\beta}{c}\frac{\mu_0 R}{2}NI_0, & R < R_0 \end{cases} \quad (14)$$

eqn. (3) means that at a given moment t' in frame F' , different z' corresponds to different t in frame F . When eqn. (13) is applied, B'_z at different z' should be computed with B at corresponding t . With eqn. (11), the magnitude of the axial field is

$$\begin{aligned} |B'_z| &= (\gamma - \frac{\gamma^2}{\gamma+1}\beta^2)|B| \\ &= \begin{cases} 0, & R > R_0 \\ \gamma(t' + \frac{\beta z'}{c})\mu_0 NI_0, & R < R_0 \end{cases} \end{aligned} \quad (15)$$

eqn. (15) shows that, spatially uniform and timely varying B in frame F becomes spatially uneven in frame F' .

Outside the solenoid, only radial magnetic induction exists and it emanates from the solenoid as if there was a magnetic line charge at z axis, i.e. the field is identical to the field computed by $\nabla' \cdot \mathbf{B}' = -\gamma\beta\mu_0\pi R_0^2NI_0c^{-1}\delta(x)\delta(y)$. Nevertheless, as to be shown, the total field flux through a closed surface vanishes when the field inside the solenoid is taken into account.

The cylindrical closed surface in fig. 3 has end surface S_1 at z'_1 and S_2 at z'_2 in frame F' , and side surface S_3 , $L = z'_2 - z'_1$ is the distance between the 2 end surfaces in frame F' . The total flux of magnetic induction through the closed surface is

$$\begin{aligned} \oint_S \mathbf{B}' \cdot \mathbf{n} da &= \int_{S_1} \mathbf{B}' \cdot \mathbf{n}_1 da + \int_{S_2} \mathbf{B}' \cdot \mathbf{n}_2 da \\ &+ \int_{S_3} \mathbf{B}' \cdot \mathbf{n}_3 da \end{aligned} \quad (16)$$

To compute the above surface integral, we first explore a special case of closed cylindrical surface with circular cross section, then present a generic derivation via Stoke's theorem.

For cylindrical surface with circular cross section of radius of R , only the radial component matters on the side surface, with eqn. (14)

$$\begin{aligned} \int_{S_3} \mathbf{B}' \cdot \mathbf{n}_3 da &= 2\pi RL|B'| \\ &= \begin{cases} -\frac{\gamma\beta\mu_0}{c}\pi R_0^2LN I_0, & R > R_0 \\ -\frac{\gamma\beta\mu_0}{c}\pi R^2LN I_0, & R < R_0 \end{cases} \end{aligned} \quad (17)$$

For the end surface, it only deals with the axial component. However, it is worth noting that in frame F' the magnetic induction at surfaces S_1 and S_2 are different.

In particular, z'_1, t' in frame F' corresponds to z_1, t_1 in frame F ,

$$\begin{aligned} t_1 &= \gamma(t' + \frac{\beta z'_1}{c}) \\ z_1 &= \gamma(z'_1 + ct'\beta) \end{aligned} \quad (18)$$

and z'_2, t' corresponds to z_2, t_2 ,

$$\begin{aligned} t_2 &= \gamma(t' + \frac{\beta z'_2}{c}) \\ z_2 &= \gamma(z'_2 + ct'\beta) \end{aligned} \quad (19)$$

then

$$t_2 - t_1 = \frac{\gamma\beta}{c}(z'_2 - z'_1) = \frac{\gamma\beta}{c}L \quad (20)$$

When computing fields in frame F' with eqn. (2), $\mathbf{B}'$ at surface S_1 shall be computed with $\mathbf{B}$ at t_1 in frame F , while $\mathbf{B}'$ at surface S_2 shall be computed with $\mathbf{B}$ at t_2 in frame F .

$$\begin{aligned} &\int_{S_1} \mathbf{B}' \cdot \mathbf{n}_1 da + \int_{S_2} \mathbf{B}' \cdot \mathbf{n}_2 da \\ &= \int_{S_2} ([B'_z]_{z'_2, t'} - [B'_z]_{z'_1, t'}) da \\ &= (\gamma - \frac{\gamma^2}{\gamma+1}\beta^2) \int_{S_2} (B(t_2) - B(t_1)) da \\ &= \int_{S_2} (B(t_2) - B(t_1)) da \end{aligned} \quad (21)$$

with eqn. (11), B vanishes outside the solenoid where $R > R_0$, so for surface S_2 with radius $R > R_0$, the above integral should be computed only on the part inside the solenoid (i.e. the area of radius $R = R_0$). For surface S_2 with radius $R < R_0$, the above integral should be computed on the whole S_2 (i.e. the area of radius R). Along with (20)

$$\begin{aligned} &\int_{S_1} \mathbf{B}' \cdot \mathbf{n}_1 da + \int_{S_2} \mathbf{B}' \cdot \mathbf{n}_2 da \\ &= \begin{cases} \frac{\gamma\beta\mu_0}{c} \pi R_0^2 L N I_0, & R > R_0 \\ \frac{\gamma\beta\mu_0}{c} \pi R^2 L N I_0, & R < R_0 \end{cases} \end{aligned} \quad (22)$$

Then with eqn. (16) and (17), the total flux of magnetic induction through the closed surface in frame F' is

$$\oint_S \mathbf{B}' \cdot \mathbf{n} da = 0 \quad (23)$$

The above derivation is a special case of a general one via Stoke's theorem. In fig. 3, the boundary $\mathbf{r}$ of the end surfaces S_2 is the intersection of S_2 and S_3 , and is closed loop. For the side surface, only the radial component $[\mathbf{B}'_{\mathbf{r}}]_{\mathbf{x}', t'}$ matters, and

$$\mathbf{n}_3 da = d\mathbf{r} \times \hat{\mathbf{z}} dz' \quad (24)$$

then

$$\int_{S_3} \mathbf{B}' \cdot \mathbf{n}_3 da = \int_L \oint_{\mathbf{r}} ([-\frac{\gamma\beta}{c} \times \mathbf{E}]_{\mathbf{x}, t}) \cdot (d\mathbf{r} \times \hat{\mathbf{z}} dz') \quad (25)$$

with identity $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$ and $\mathbf{E} \cdot \hat{\mathbf{z}} = 0$, $\beta \cdot \hat{\mathbf{z}} = \beta$

$$\int_{S_3} \mathbf{B}' \cdot \mathbf{n}_3 da = \int_L \oint_{\mathbf{r}} (\frac{\gamma\beta}{c} [\mathbf{E}]_{\mathbf{x}, t}) \cdot d\mathbf{r} dz' \quad (26)$$

with Stokes' theorem on the closed loop $\mathbf{r}$

$$\int_{S_3} \mathbf{B}' \cdot \mathbf{n}_3 da = \int_L \int_{S_2} -\frac{\gamma\beta}{c} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}_2 da dz' \quad (27)$$

since S_2 is perpendicular to z axis, $\mathbf{n}_2 = \hat{\mathbf{z}}$, then

$$\int_{S_3} \mathbf{B}' \cdot \mathbf{n}_3 da = \int_{S_2} \int_L -\frac{\gamma\beta}{c} \frac{\partial \mathbf{B}_z}{\partial t} \cdot \hat{\mathbf{z}} dz' da \quad (28)$$

at a given moment t' in frame F' , with Lorentz transformation, one can have the differential form of eqn. (20)

$$dt = \frac{\beta\gamma}{c} dz' \quad (29)$$

along with (13)

$$\frac{\gamma\beta}{c} \frac{\partial \mathbf{B}_z}{\partial t} = \frac{\partial \mathbf{B}_z}{\partial z'} = \frac{\partial \mathbf{B}'_z}{\partial z'} \quad (30)$$

then

$$\begin{aligned} \int_{S_3} \mathbf{B}' \cdot \mathbf{n}_3 da &= \int_{S_2} \int_L -\frac{\partial \mathbf{B}'_z}{\partial z'} \cdot \hat{\mathbf{z}} dz' da \\ &= \int_{S_2} (\mathbf{B}'_z(z'_1, t') - \mathbf{B}'_z(z'_2, t')) \cdot \hat{\mathbf{z}} da \end{aligned} \quad (31)$$

to derive $\mathbf{B}'_z$ with eqn. (13), $\mathbf{B}'_z$ at z'_1 shall be computed with $\mathbf{B}$ at t_1 in frame F , while $\mathbf{B}'_z$ at z'_2 shall be computed with $\mathbf{B}$ at t_2 in frame F . So in general, the radial field flux through side surface is nonvanishing.

For the end surface, $\mathbf{n}_2 = \hat{\mathbf{z}}$, $\mathbf{n}_1 = -\hat{\mathbf{z}}$

$$\begin{aligned} &\int_{S_1} \mathbf{B}' \cdot \mathbf{n}_1 da + \int_{S_2} \mathbf{B}' \cdot \mathbf{n}_2 da \\ &= \int_{S_2} (\mathbf{B}'_z(z'_2, t') - \mathbf{B}'_z(z'_1, t')) \cdot \hat{\mathbf{z}} da \end{aligned} \quad (32)$$

Then with eqn. (16) and (31), the total flux of magnetic induction through the closed surface in frame F' is

$$\oint_S \mathbf{B}' \cdot \mathbf{n} da = 0 \quad (33)$$

Thus, in the moving frame F' , the axial field flux through end surface cancels out the radial field flux through side surface, and the total flux of magnetic induction $\mathbf{B}'$ through a closed surface vanishes. Nevertheless, the axial field exists only inside the solenoid, and

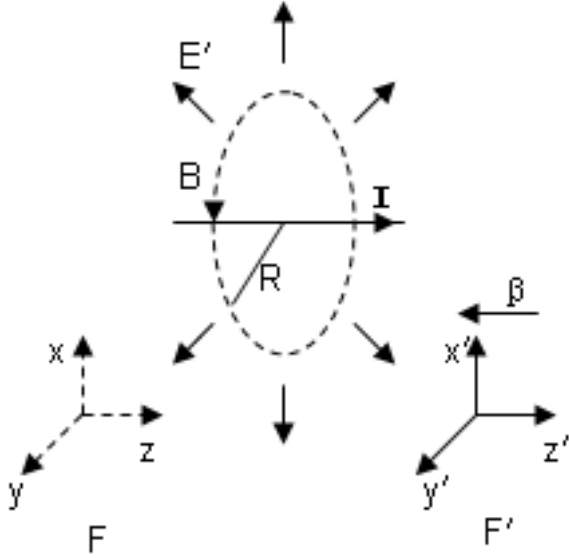


FIG. 4: Fields observed around a straight wire carrying static current I . Dashed line shows that in frame F , the force line of magnetic induction $\mathbf{B}$ is concentric circular. Solid lines are electric field $\mathbf{E}'$ observed in frame F' moving with velocity β relative to frame F , and $\mathbf{E}'$ has only radial component.

eqn. (14) shows, outside the solenoid only the radial magnetic induction exists and it is indistinguishable as if there was magnetic line charge inside the solenoid.

The above scenario of solenoid has a dual case with a long straight wire carrying a static current I , unlike the solenoid case in which the axial field flux cancels out the radial field flux, the long straight wire can appear having non-zero divergence under relativistic transformation. Around a long straight wire carrying a static current I , the force lines of magnetic induction $\mathbf{B}$ are concentric circular, as the dashed lines shown in fig. 4. The magnitude of $\mathbf{B}$ is [14]

$$|\mathbf{B}| = \frac{\mu_0}{2\pi R} I \quad (34)$$

where R is the perpendicular distance from the observation point to the wire. When viewed from a frame F' moving along z axis, an electric field $\mathbf{E}'$ emerges, and by eqn. (2) its magnitude can be computed as

$$|\mathbf{E}'| = c\gamma\beta \frac{\mu_0}{2\pi R} I \quad (35)$$

Since $\mathbf{B}$ is loop shaped, $\mathbf{E}'$ is radial. This radial electric field induced by concentric circular shaped magnetic induction forms a realization of the field transformation shown in fig. 1b.

To compute the flux of field $\mathbf{E}'$ through a closed cylindrical surface S around the straight wire, we notice that $\mathbf{E}'$ has only radial component, so the integral is calculated only on the side surface, then

$$\oint_S \mathbf{E}' \cdot \mathbf{n} da = 2\pi RL|\mathbf{E}'| = c\gamma\beta\mu_0 LI \quad (36)$$

the field flux in eqn. (36) varies with observing frame and can be nonvanishing even without net charge.

The divergence of the electric field $\mathbf{E}'$ in Frame F' is

$$\begin{aligned} \nabla' \cdot [\mathbf{E}']_{\mathbf{x}',t'} &= \nabla' \cdot [(\gamma(\mathbf{E} + c\beta \times \mathbf{B}) - \frac{\gamma^2}{\gamma+1}\beta(\beta \cdot \mathbf{E}))]_{\mathbf{x},t} \quad (37) \end{aligned}$$

Around the straight wire, $\mathbf{E} = 0$ in frame F , then from eqn. (2)

$$[\mathbf{E}']_{\mathbf{x}',t'} = [\gamma(c\beta \times \mathbf{B})]_{\mathbf{x},t} \quad (38)$$

Since β is along the z axis, the z components of field $B_z = 0, E'_z = 0$, eqn. (42) becomes

$$\nabla' \cdot [\mathbf{E}']_{\mathbf{x}',t'} = \nabla \cdot [\mathbf{E}']_{\mathbf{x}',t'} = \nabla \cdot [\gamma(c\beta \times \mathbf{B})]_{\mathbf{x},t} \quad (39)$$

along with $\nabla \times \beta = 0$

$$\nabla' \cdot [\mathbf{E}']_{\mathbf{x}',t'} = -[c\gamma\beta \cdot \nabla \times \mathbf{B}]_{\mathbf{x},t} \quad (40)$$

For the setup in fig. 4, magnetic induction $\mathbf{B}$ has only x and y components, and $\mathbf{J} = \hat{\mathbf{z}}I\delta(x)\delta(y)$, with $\hat{\mathbf{z}}$ being the unit z vector. Thus

$$\nabla \times [\mathbf{B}]_{\mathbf{x},t} = \mu_0 \mathbf{J} = \mu_0 \hat{\mathbf{z}}I\delta(x)\delta(y) \quad (41)$$

then

$$\nabla' \cdot [\mathbf{E}']_{\mathbf{x}',t'} = -c\gamma\beta\mu_0 I\delta(x)\delta(y) \quad (42)$$

Though $\mathbf{E} = 0$ thus $\nabla \cdot \mathbf{E} = 0$ in frame F , for $\beta \neq 0$ the divergence of $\mathbf{E}'$ observed in frame F' is none zero at z axis, which is equivalent to a line charge at z axis.

The straight wire and the solenoid discussed above are similar in a couple of aspects. 1) The linearly varying axial magnetic field inside the solenoid creates a static magnetic displacement current J_m , in analogue to the electric displacement current. Thus the solenoid carrying a linearly varying electric current can be viewed as a static magnetic current in a straight wire. 2) In both cases, radial field component emanating from the axis can be observed from an axially moving frame F' , as if there was a line charge at the axis.

On the other hand, in the solenoid setup, the evenly distributed, linearly varying axial field component inside the solenoid becomes uneven when viewed from an axially moving frame F' . This unevenness contributes to the axial field flux to cancel out the flux of the radial field component. But in the straight wire case, such axial field component does not exist, therefore, the observed radial electric field $\mathbf{E}'$ has varying divergence in different observing frame, as shown in eqn. (42), which violates the Gauss law of Maxwell equations.

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